Università degli Studi di Padova

Scalar and Predictive Quantization

Marco Cagnazzo

Multimedia Communications

Outline

Definitions

Uniform Quantization

Examples

Rate-distortion curve

Optimal Scalar Quantization

Predictive scalar quantization

Outline

Definitions

Uniform Quantization

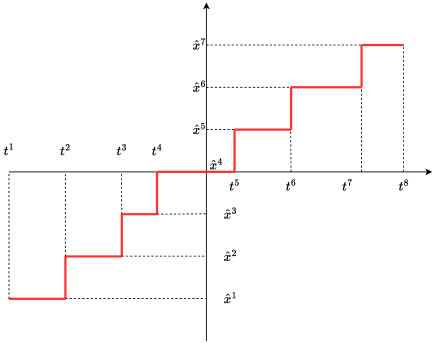
Examples

Rate-distortion curve

Optimal Scalar Quantization

Predictive scalar quantization

Scalar quantization (SQ): it is just a staircase function (almost always)

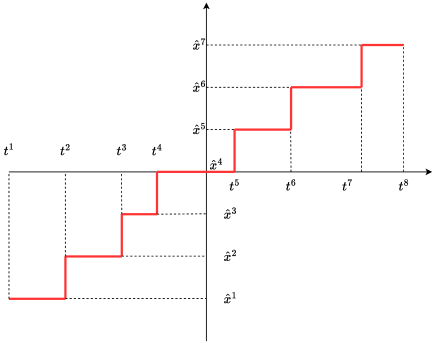


Here we show the graph of a possible $Q(x)$

Go to **wooclap.com** with the code **PBTFUO**

Input range:

Scalar quantization (SQ): it is just a staircase function (almost always)



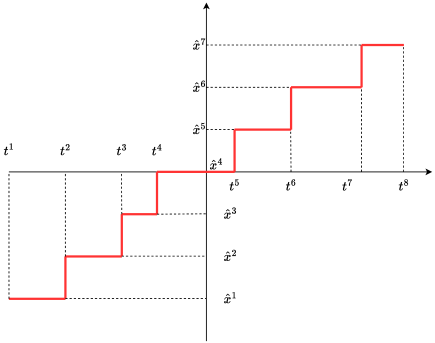
Here we show the graph of a possible $Q(x)$

Go to **wooclap.com** with the code **PBTFUO**

Input range: (t_1, t_8)

Number of levels:

Scalar quantization (SQ): it is just a staircase function (almost always)



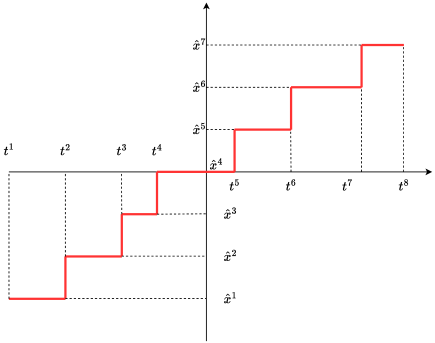
Here we show the graph of a possible $Q(x)$

Go to **wooclap.com** with the code **PBTFUO**

Input range: (t_1, t_8)

Number of levels: $L = 7$

Scalar quantization (SQ): it is just a staircase function (almost always)



Here we show the graph of a possible $Q(x)$

Go to **wooclap.com** with the code **PBTFUO**

Input range: (t_1, t_8)

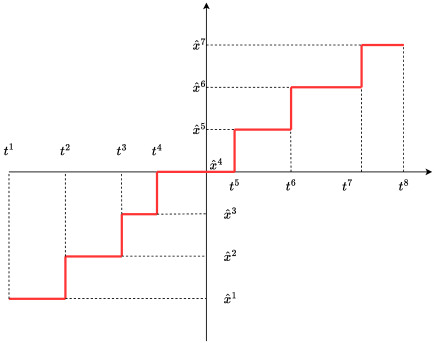
Number of levels: $L = 7$

The t^i's determine a *partition* of the input interval

Each $\Theta^i = (t^i, t^{i+1})$ is a quantization region or cell

Any number in $\Theta^i = (t^i, t^{i+1})$ is quantized as $\hat{x}^i$

Scalar quantization (SQ): it is just a staircase function (almost always)



Here we show the graph of a possible $Q(x)$

Go to **wooclap.com** with the code **PBTFUO**

Input range: (t_1, t_8)

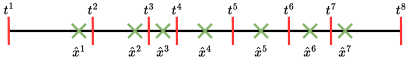
Number of levels: $L = 7$

The t^i's determine a *partition* of the input interval

Each $\Theta^i = (t^i, t^{i+1})$ is a quantization region or cell

Any number in $\Theta^i = (t^i, t^{i+1})$ is quantized as $\hat{x}^i$

The graph below also represent the quantizer: the thresholds and the levels competely determine the quantizer



Definition : scalar quantization (SQ)

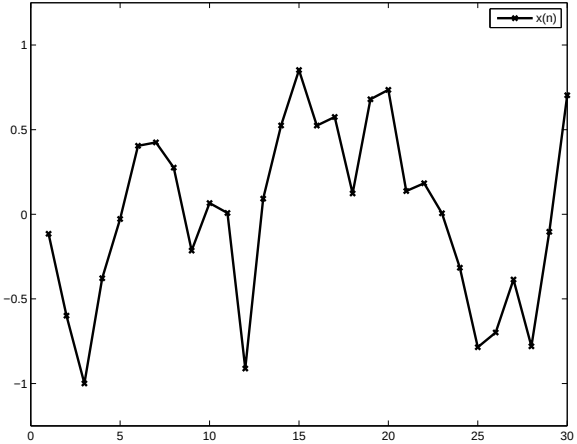
Scalar quantization (SQ) is a function Q from $\mathbb{R}$ to a discrete set called *Dictionary*

$$Q : x \in \mathbb{R} \rightarrow y \in \mathcal{C} = \{\hat{x}^1, \hat{x}^2, \dots \hat{x}^L\} \subset \mathbb{R}$$

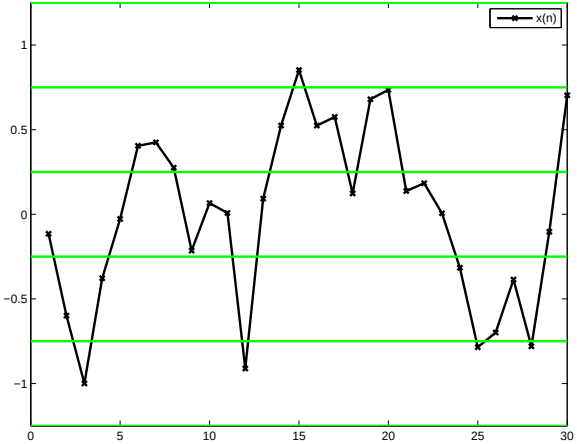
- C : Dictionary, it is a discrete subset of $\mathbb{R}$
- $\hat{x}^i$: quantization level, code-word
- $e = x - Q(x)$: Quantization noise
- $\Theta^i = \{x : Q(x) = \hat{x}^i\}$: Decision regions or cells

Regions and levels completely define the SQ
Regions are typically defined as intervals: $\Theta^i = (t^i, t^{i+1})$
Therefore, thresholds and levels also completely define a SQ

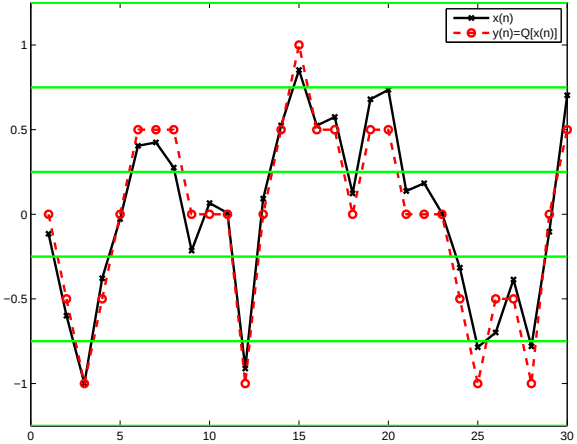
Example 1



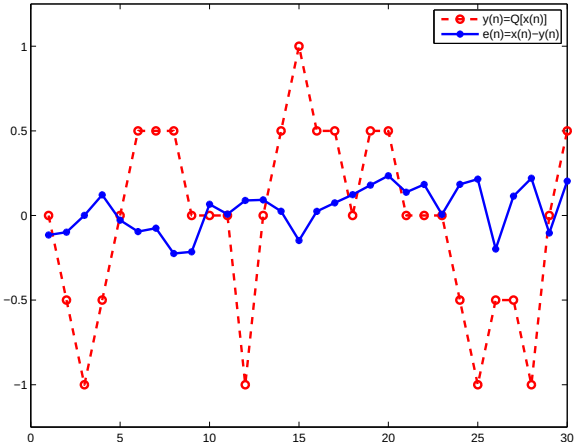
Example 1



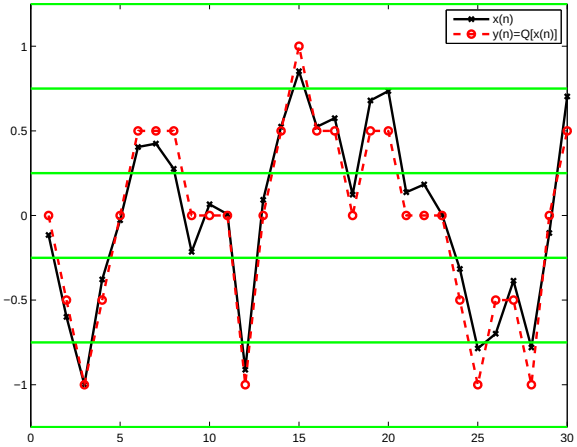
Example 1



Example 1



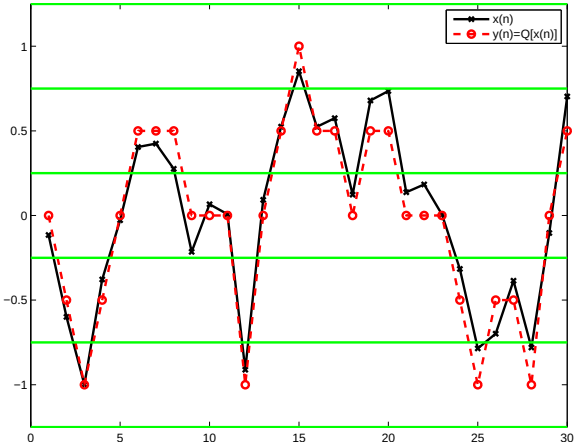
Example 1



Go to **wooclap.com** with the code **PBTFUO**

Input interval:

Example 1

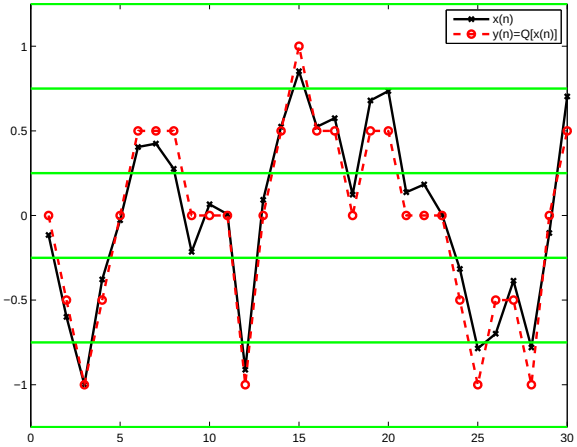


Go to **wooclap.com** with the code **PBTFUO**

Input interval: $(-A, A) = (-1.25, 1.25)$

Thresholds:

Example 1

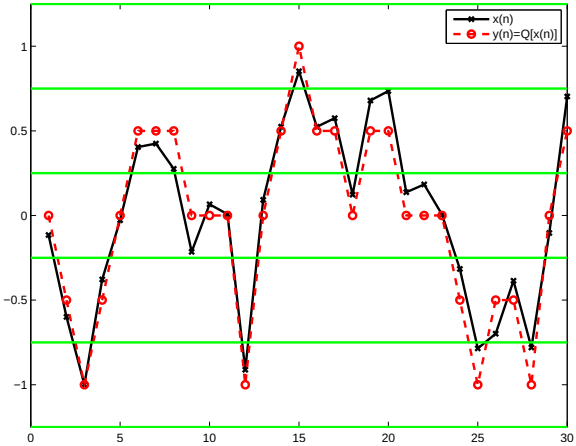


Go to **wooclap.com** with the code **PBTFUO**

Input interval: $(-A, A) = (-1.25, 1.25)$

Thresholds: $-1.25, -0.75, -0.25, 0.25, 0.75, 1.25$

Example 1



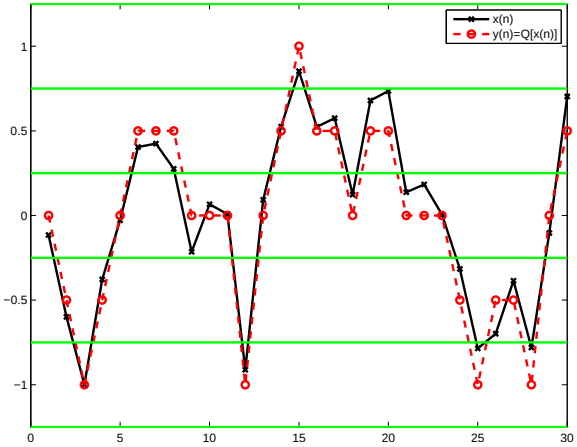
Go to **wooclap.com** with the code **PBTFUO**

Input interval: $(-A, A) = (-1.25, 1.25)$

Thresholds: $-1.25, -0.75, -0.25, 0.25, 0.75, 1.25$

Levels:

Example 1



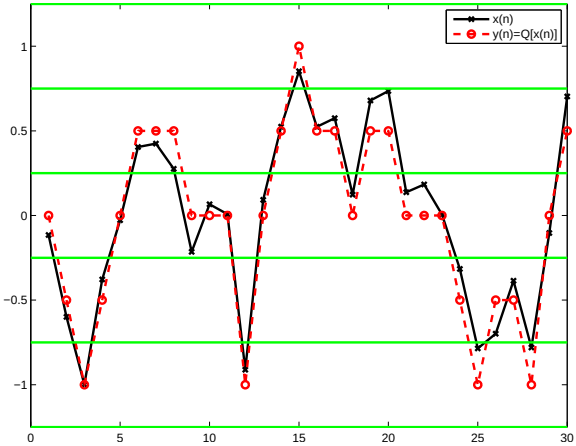
Go to **wooclap.com** with the code **PBTFUO**

Input interval: $(-A, A) = (-1.25, 1.25)$

Thresholds: $-1.25, -0.75, -0.25, 0.25, 0.75, 1.25$

Levels: $-1, -0.5, 0, 0.5, 1$

Example 1



Go to **wooclap.com** with the code **PBTFUO**

Input interval: $(-A, A) = (-1.25, 1.25)$

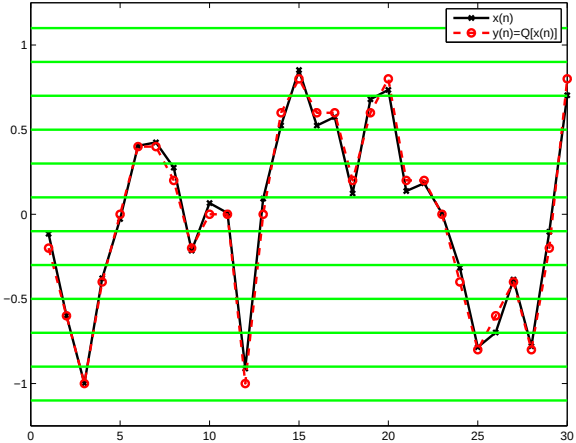
Thresholds: $-1.25, -0.75, -0.25, 0.25, 0.75, 1.25$

Levels: $-1, -0.5, 0, 0.5, 1$

Note that:

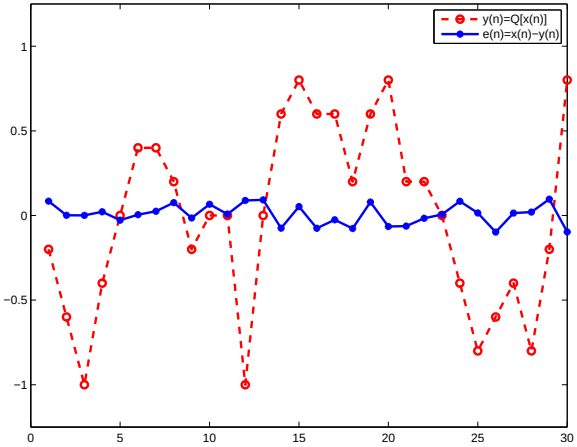
- All the cells have the same size $\Delta = \frac{2A}{L} = 2.5/5 = 0.5$
- Each level is the center of the corresponding cell

Example 2



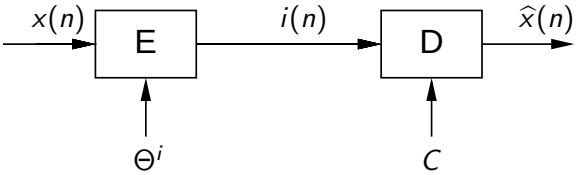
Here we have again cells with identical size and levels at the center of the cell
The quantization cells are smaller $\Delta = 0.2$
The quantization error is typically smaller

Example 2



Here we have again cells with identical size and levels at the center of the cell
The quantization cells are smaller $\Delta = 0.2$
The quantization error is typically smaller

Quantization as coding / decoding



- The quantization process can be seen as the cascade of "encoding" and "decoding"
- The encoder produces the index of the quantization region, $i(n) \in \mathbb{Z}$
- The number $i(n)$ is represented with a suitable string of bits, e.g by using *lossless coding*
- The decoder associates to $i(n)$ to the corresponsing level (or code-word) $\widehat{x}^i(n)$
- In a compression system, often with the term quantization we just refer to the operation $x \to i$, while the evaluation of the quantization level, $i \to \widehat{x}^i$ is referred to as *Inverse Quantization*
  - The first is performed at the encoder, the second at the decoder

Rate of a scalar quantizer

- **Rate** R and **distortion** D are the performance criteria of a quantizer
- The rate of a scalar quantizer is the average number of bits needed to represent a sample, or rather, its quantization index: it amounts thus to the binarization of $i(n)$
- It is *always done via lossless coding*
- We will assume $R = \log_2 L$
- This is a good approximation of real-life coding rate, with two implicit assumption

Rate of a scalar quantizer

- **Rate** R and **distortion** D are the performance criteria of a quantizer
- The rate of a scalar quantizer is the average number of bits needed to represent a sample, or rather, its quantization index: it amounts thus to the binarization of $i(n)$
- It is *always done via lossless coding*
- We will assume $R = \log_2 L$
- This is a good approximation of real-life coding rate, with two implicit assumption
  - Quantized data have a uniform probability distribution (pessimistic)
  - The binary representation of $i(n)$ is the best (entropy coder)
- We will elaborate on this problem in the next lesson

Distortion of a scalar quantizer

- We use the squared error, which, on a single sample is:

$$d\left[x(n), Q[x(n)]\right] = |e(n)|^2 = |x(n) - Q[x(n)]|^2$$

- For a signal $x(\cdot)$ of duration N, we use the mean square error (MSE):

$$D = \frac{1}{N}\sum_{n=0}^{N-1} d[x(n), Q[x(n)]]$$

- For random signals[1], $D = \mathrm{E}\left\{|X(n) - Q(X(n))|^2\right\} = \mathrm{E}\left\{|E(n)|^2\right\}$
- In this case, distortion is the variance of the random process $E(n) = X(n) - Q(X(n))$, and is indicated as σ_{Q}^2

[1] Notice that we use lower-case letters for deterministic signals and upper-case letter for random variables/signals, while E is the expectation operator.

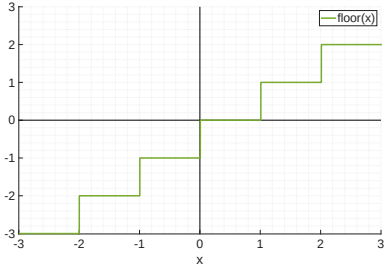
Mathematical reminders: integer mapping

- Let us remind the integer mapping functions commonly used in programming: `floor`, `ceil`, `round`, `fix`
- They are "staircase functions"
- They are widely used to implement quantizers

Mathematical Reminders: Floor Function

Operator: $\lfloor x \rfloor$

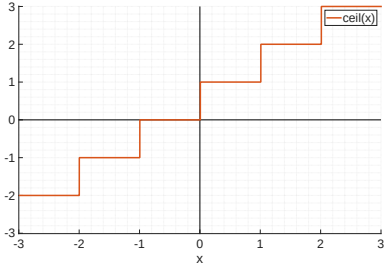
- **Definition**: The greatest integer less than or equal to x
- **Behavior**: Always rounds towards $-\infty$
- **Examples**:
  - $\lfloor 2.7 \rfloor = 2$
  - $\lfloor -2.7 \rfloor = -3$
- **Usage**: Common in basic uniform quantization models



Mathematical Reminders: Ceil Function

Operator: $\lceil x \rceil$

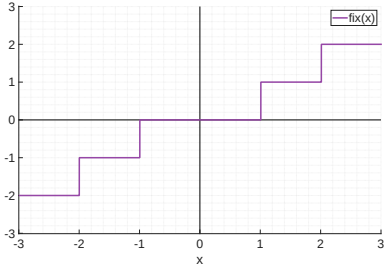
- **Definition**: The smallest integer greater than or equal to x
- **Behavior**: Always rounds towards $+\infty$
- **Examples**:
  - $\lceil 2.1 \rceil = 3$
  - $\lceil -2.1 \rceil = -2$
- **Usage**: Used for determining buffer sizes or resource allocation bounds



Mathematical Reminders: Fix Function

Operator: $\text{fix}(x)$

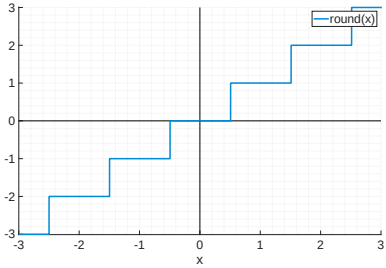
- **Definition**: Drops the fractional part of the number
- **Behavior**: Rounds towards zero (Symmetric)
  - Equivalent to $\lfloor x \rfloor$ for $x \geq 0$.
  - Equivalent to $\lceil x \rceil$ for $x < 0$.
- **Examples**:
  - $\text{fix}(2.7) = 2$
  - $\text{fix}(-2.7) = -2$
- **Note**: Standard behavior for integer casting in most programming languages



Mathematical Reminders: Round Function

Operator: round(x)

- **Definition**: Maps x to the nearest integer
- **Behavior**: Usually implemented as $\lfloor x + 0.5 \rfloor$
- **Examples**:
  - $\text{round}(2.51) = 3$
  - $\text{round}(2.49) = 2$
  - $\text{round}(-2.3) = -2$
- **Usage**: allow a straightforward implementation of *uniform quantization*



Outline

Definitions

Uniform Quantization

Examples

Rate-distortion curve

Optimal Scalar Quantization

Predictive scalar quantization

Uniform quantization

Uniform quantization is characterized by:

- The input range: $(0, A)$ (unsigned data) or $\left(-\frac{A}{2}, \frac{A}{2}\right)$ (signed data)
- The number of levels L

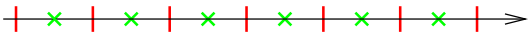
The input range is divided into L equal-sized cells. The cell size is $\Delta = \frac{A}{L}$. Each cell is represented by its mid-point, i.e. the quantization levels are the centers of the quantization cells.

$$\forall i,\ \Delta^i = \Delta = A/L$$

$$t^i = t^{i-1} + \Delta \text{ (red bars)}$$

$$\hat{x}^i = \frac{t^i + t^{i-1}}{2} \text{ (green crosses)}$$

$$\Theta^i = \left(\hat{x}^i - \frac{\Delta}{2}, \hat{x}^i + \frac{\Delta}{2}\right)$$



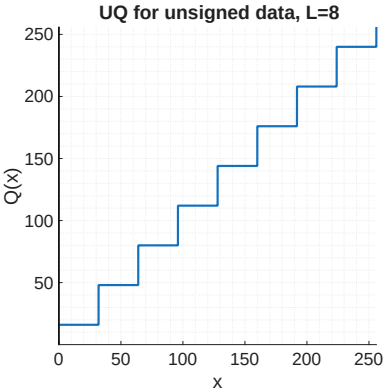
Uniform Quantization for unsigned data

For **unsigned data** (range $(0, A)$) the thresholds are $(0, \Delta, 2\Delta, \ldots, L\Delta)$, i.e., $t^i = (i-1)\Delta$.
We can thus define the behavior of the **encoder** (from x to the quantization index i), of the **decoder** (from i to the quantization level $\hat{x}^i$) and of the full quantization chain:

$$i = \left\lceil \frac{x}{\Delta} \right\rceil \qquad \text{encoder}$$

$$\hat{x}^i = i \cdot \Delta - \frac{\Delta}{2} \qquad \text{decoder}$$

$$Q(x) = \Delta \cdot \left\lceil \frac{x}{\Delta} \right\rceil - \frac{\Delta}{2}$$



Uniform quantization for signed data

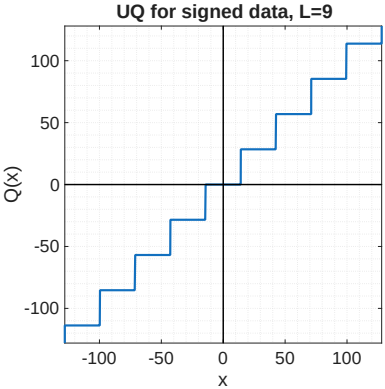
For **signed data** (range in $(-A/2, A/2)$) one typically wants zero to be a quantization level (**mid-tread** quantizer): in such a way, small fluctuations are quantized to zero. Therefore, we **always use an odd number of levels for UQ of signed data**. In this case, uniform quantization (UQ) amounts to **rounding**:

$$i = \text{round}\left(\frac{x}{\Delta}\right) \qquad \text{encoder}$$

$$\hat{x}^i = \Delta \cdot i \qquad \text{decoder}$$

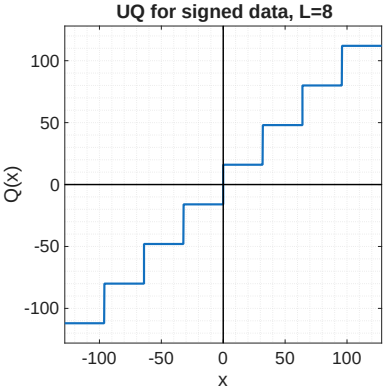
$$Q(x) = \Delta \cdot \text{round}\left(\frac{x}{\Delta}\right)$$

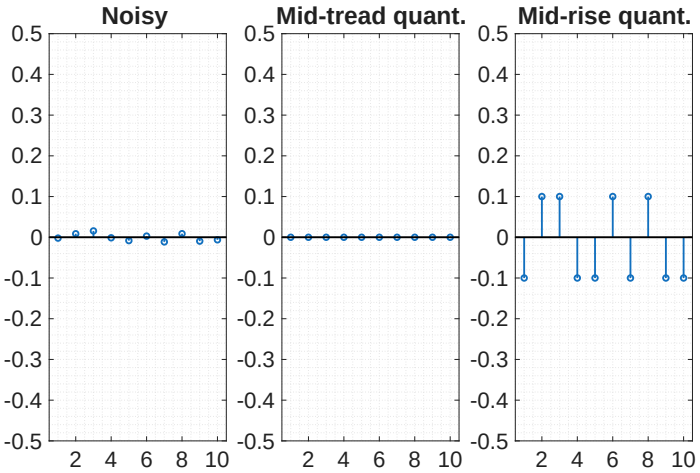
In other words, $Q(x)$ is **the closest multiple of** Δ. This quantizer is also referred to as **mid-tread** quantizer.



Uniform quantization for signed data

If one uses uniform quantization with an even number of levels for signed data, the central threshold $t_{\frac{N}{2}}$ is zero, see figure on the right.
This quantizer is also referred to as **mid-rise** quantizer.
Since zero is a threshold any very small value will be quantized as $\pm\Delta/2$ according to its sign.
This can be a problem is the signal has small fluctuations around zero due to additive noise: quantization enhances noise!





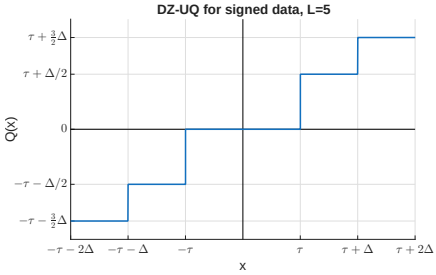
On the left, we show $x(n)$, a signal with small values that are **interpreted** as noise: uninformative, random fluctuations.
In the middle, we show $Q(x)$ when zero is a quantization level (**mid-tread** quantizer): the small values are forced to zero, making it "easier" to encode the signal.
On the right, we show $Q(x)$ when zero is a threshold (**mid-rise** quantizer): after quantization the noise is amplified instead of being removed!

Deadzone quantization

Pushing further this idea, we can design a quantizer where the central cell is larger than the others. This allows for removing more small-valued oscillations, while keeping a fine resolution on non-zero values. This is called **deadzone** quantization (DZQ) and is very commonly used in data compression.
Using the threshold value τ, the DZQ can be implemented as:

$$i = \begin{cases} \text{sign}(x) \cdot \left\lfloor \frac{|x| + \frac{\tau\Delta}{2}}{\Delta} \right\rfloor & \text{if } |x| \geq \tau \\ 0 & \text{if } |x| < \tau \end{cases} \qquad \text{encoder}$$

$$\hat{x} = \begin{cases} \text{sign}(i) \cdot \Delta \left(|i| + \frac{1-\tau}{2} \right) & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \end{cases} \qquad \text{decoder}$$



UQ: wrap-up

- UQ is intuitive: just split the input range into L intervals, and represent each interval with its middle.
- Implementation is easy:

$$Q(x) = \Delta \cdot \mathrm{round}\left(\frac{x}{\Delta}\right) \qquad \text{mid-tread}$$

$$Q(x) = \Delta \cdot \mathrm{floor}\left(\frac{x}{\Delta}\right) + \frac{\Delta}{2} \qquad \text{mid-rise}$$

- For a given range A and a given number of levels L, UQ has the smallest maximum error (optimal **minimax** quantizer)
  - The maximum error is half the size of the maximum quantization interval: any modification of the UQ increases the maximum quantization interval and thus the maximum error
- Last, but not least, UQ allows for an analytical formula of the RD (rate-distortion) curve. But before that...

Outline

Definitions

Uniform Quantization

Examples

Rate-distortion curve

Optimal Scalar Quantization

Predictive scalar quantization

Image quantization: example 1

Image quantization: example 1

Rate 21 bpp PSNR 47.19 dB Compression ratio 1.143

Image quantization: example 1

Rate 18 bpp PSNR 42.38 dB Compression ratio 1.333

Image quantization: example 1

Rate 15 bpp PSNR 36.97 dB Compression ratio 1.600

Image quantization: example 1

Rate 12 bpp PSNR 31.40 dB Compression ratio 2.000

Image quantization: example 1

Rate 9 bpp PSNR 29.26 dB Compression ratio 2.667

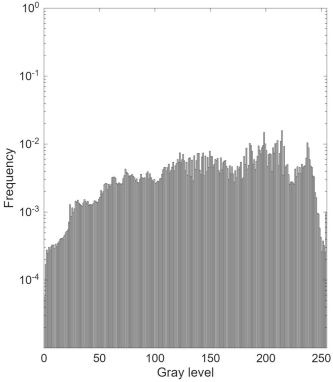
Image quantization: example 1

Rate 6 bpp PSNR 27.83 dB Compression ratio 4.000

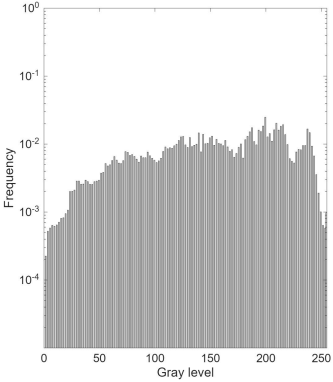
Image quantization: example 1

Rate 3 bpp PSNR 25.75 dB Compression ratio 8.000

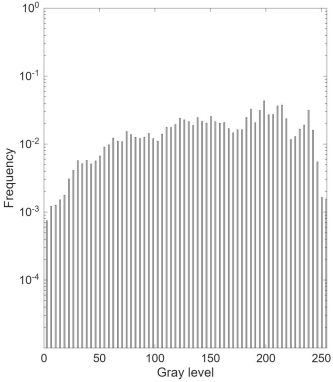
Scalar quantization: example 2



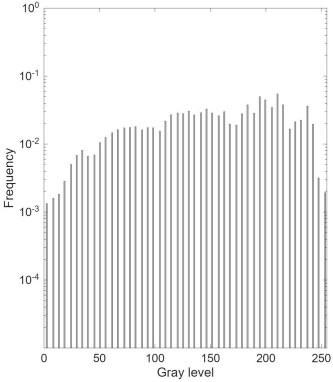
Scalar quantization: example 2



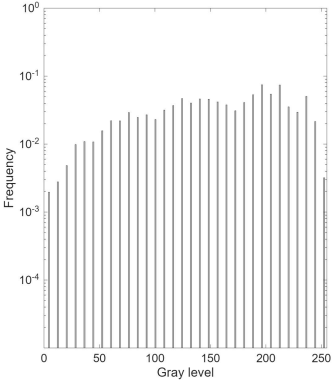
Scalar quantization: example 2



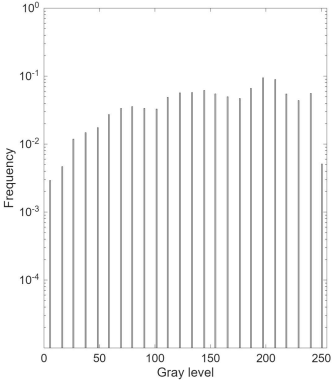
Scalar quantization: example 2



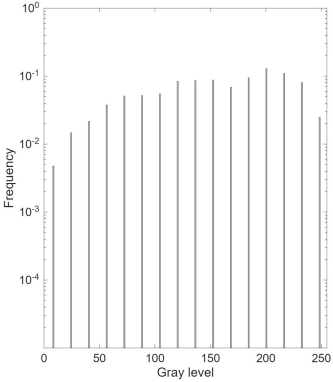
Scalar quantization: example 2



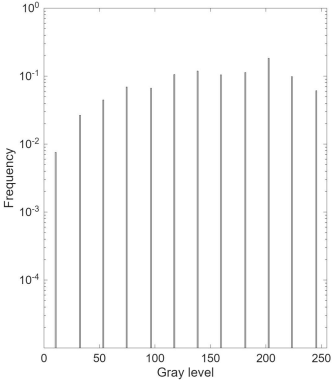
Scalar quantization: example 2



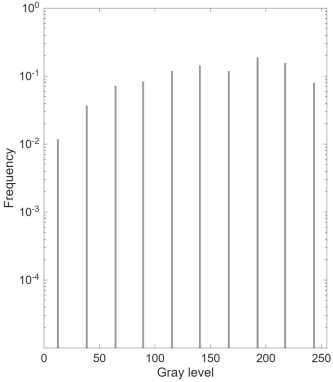
Scalar quantization: example 2



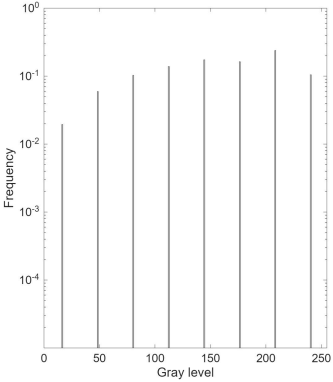
Scalar quantization: example 2



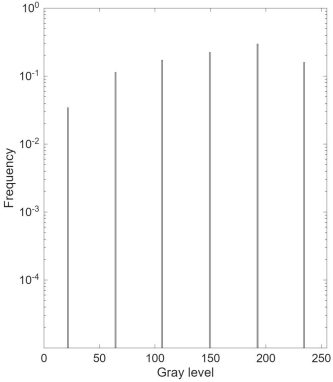
Scalar quantization: example 2



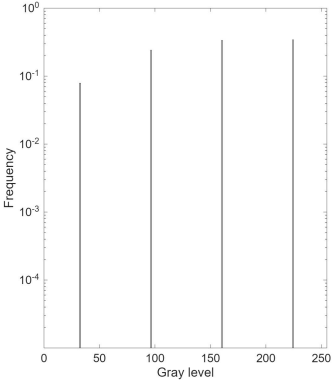
Scalar quantization: example 2



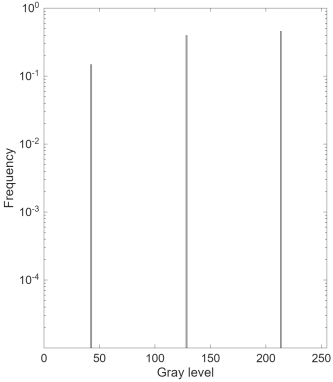
Scalar quantization: example 2



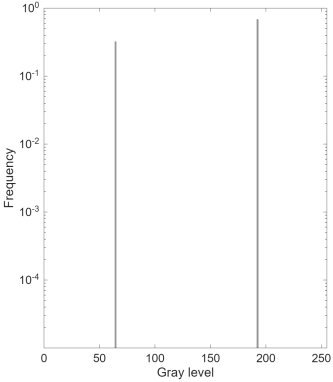
Scalar quantization: example 2



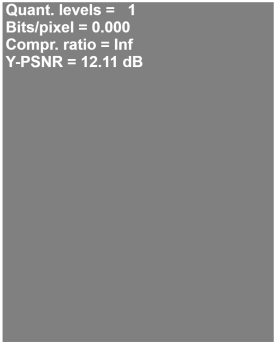
Scalar quantization: example 2

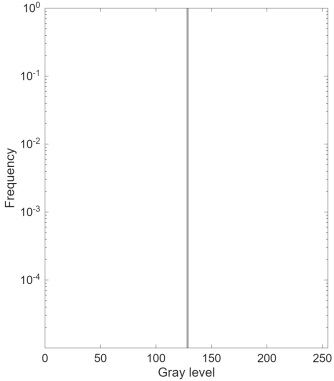


Scalar quantization: example 2

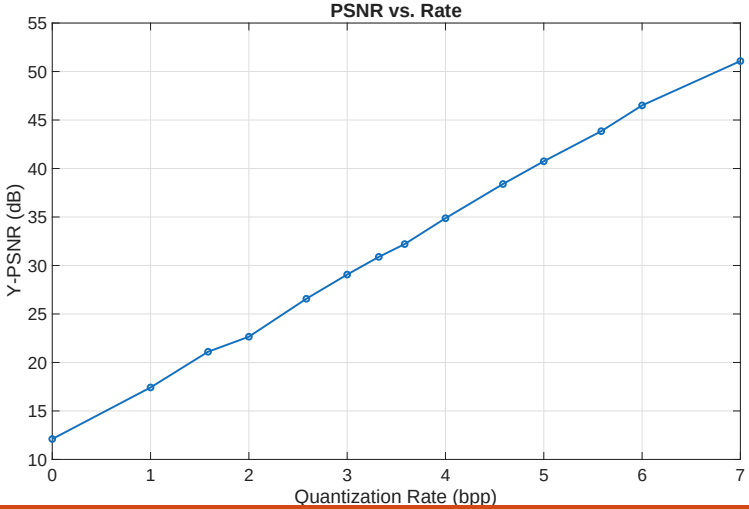


Scalar quantization: example 2

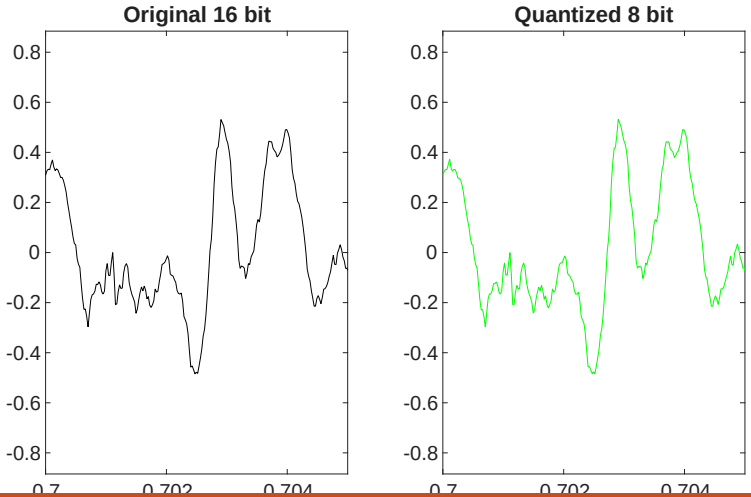




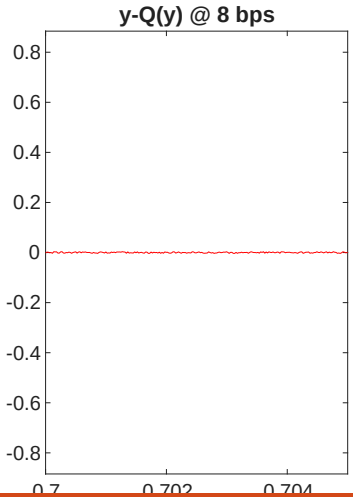
Scalar quantization: example 2

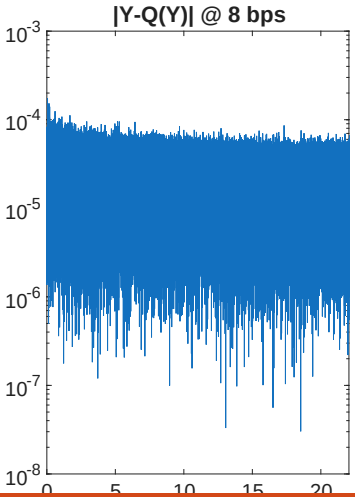


Audio quantization example

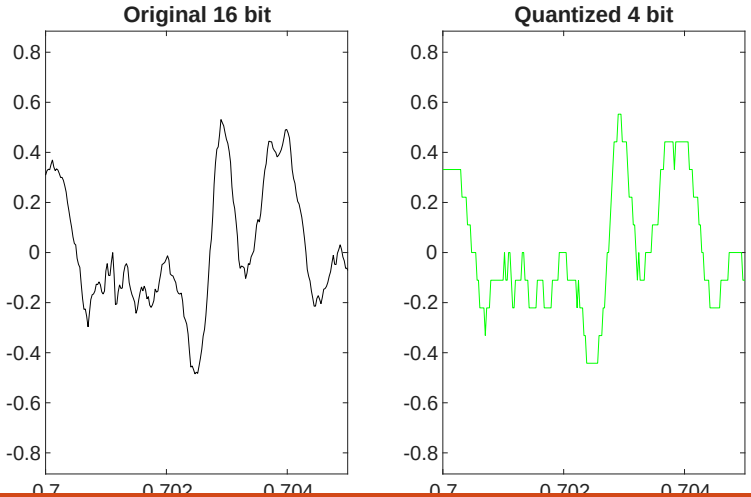


Audio quantization example

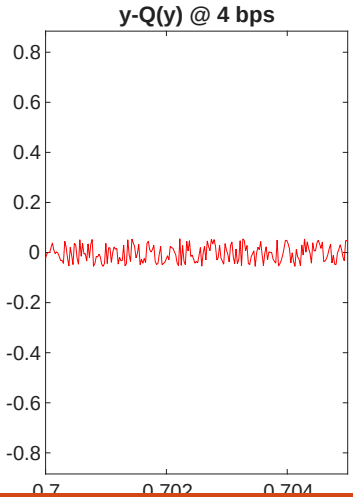


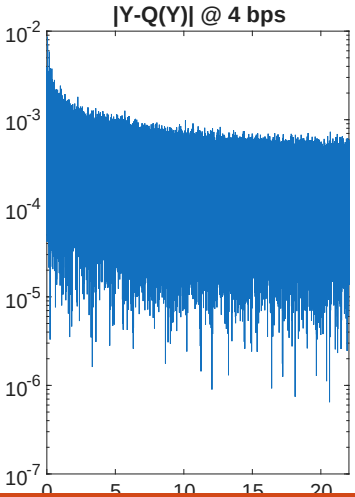


Audio quantization example

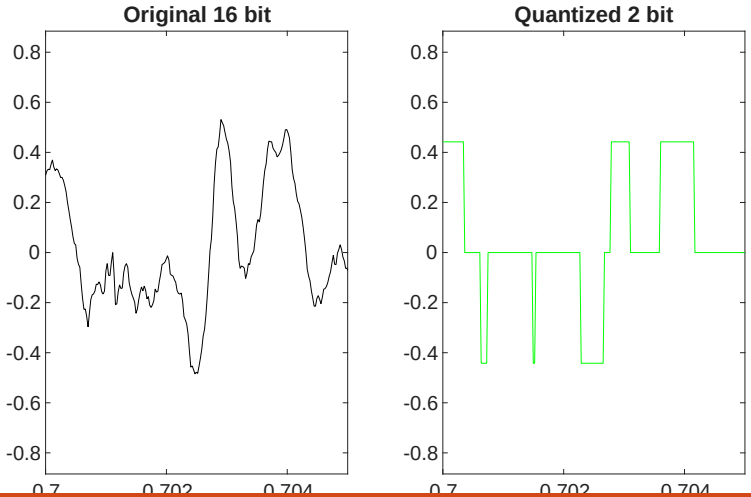


Audio quantization example

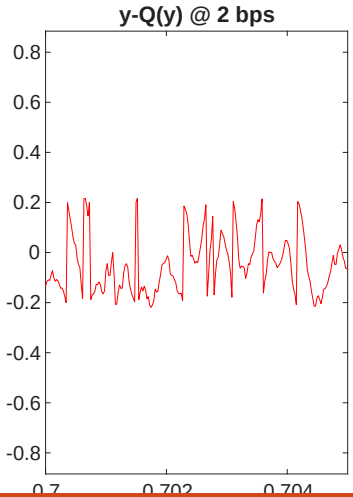


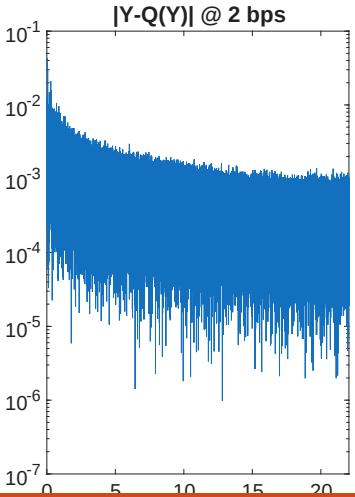


Audio quantization example



Audio quantization example





Audio Quantization: Subjective Comparison

Reference	Medium Rate	Low Rate	Very Low Rate
Original (16 bits)	Quantized (8 bits)	Quantized (4 bits)	Quantized (2 bits)

- **8-bit**: Notice the slight "hissing" or granular noise.
- **4-bit**: Observe the significant loss of fidelity and prominent quantization noise
- **2-bit**: The noise is stron; however the speech can still be understood, while the music is totally lost

Outline

Definitions

Uniform Quantization

Examples

Rate-distortion curve

Optimal Scalar Quantization

Predictive scalar quantization